Analysis of IMO medal boundary choices

Joseph Myers

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Introduction

Medal boundaries at the International Mathematical Olympiad are determined by the following from the General Regulations:

> **5.6** The total number of prizes (first, second and third) must be approved by the Jury and must not exceed half the total number of Contestants unless this is approved by at least two thirds of the members of the Jury. The numbers of first, second and third prizes must be approximately in the ratio 1 : 2 : 3.

This was agreed in 2016 (with effect from 2017, but with the substance of the changes having been applied in 2015 and 2016 via the Annual Regulations), replacing the following:

> **5.6** The total number of prizes (first, second and third) must be approved by the Jury and should not normally exceed half the total number of Contestants. The numbers of first, second and third prizes must be approximately in the ratio 1 : 2 : 3.

This was a revision of previous wording used in 2011 and 2012, the regulations of which IMOs were used as the basis for the General Regulations:

> **5.6** The total number of prizes (first, second and third) will not exceed half the total number of Contestants. The numbers of first, second and third prizes will be approximately in the ratio 1 : 2 : 3.

The principles date back at least as far as IMO 1984 (whose regulations appear in *International Mathematical Olympiads 1978–1985 and Forty Supplementary Problems* by Murray S. Klamkin (MAA)):

> The total number of awarded prizes will not exceed half of the number of all contestants. The number of 1st, 2nd, and 3rd prizes awarded will if possible be in the ratio 1 : 2 : 3.

In practice, first, second and third prizes are generally known as gold, silver and bronze medals. They are awarded based on ranking contestants by total score, and as many medals in total are awarded as possible consistent with not giving more than half of contestants medals. On several occasions, before the introduction of the 2/3-majority requirement, more than half of contestants have been given medals where this seemed fairer to the Jury in light of the particular distribution of scores at that IMO, leading to the insertion of "normally" to reflect existing practice when the regulations were split into General and Annual Regulations, with revisions of the General Regulations being the responsibility of the Jury instead of being decided by each host country.

At IMO 2015, changes were debated to make it harder to choose to give more than half the contestants medals, with the result that it was agreed (via a change to the Annual Regulations approved by the IMO Advisory Board for 2015 only) that giving more than half the contestants medals would require a 2/3 majority; this was applied via the Annual Regulations again in 2016, and was adopted permanently from 2017. In addition, a new procedure was introduced in 2015 where the medal boundaries were debated and voted on based on figures and bar charts for the numbers of medals of each type, without information about the scores to which those corresponded, to make it less likely that votes would be based on self-interest. However, the debate about the merits of different boundaries still continued for a long time.

This document analyses various possible algorithms for deciding medal boundaries based on how well they agree with past decisions made by the Jury, with the idea that it would be fairer if a consistent algorithm were used to decide medal boundaries (whether always, or unless the Jury decides by a super-majority that the results of the algorithm are clearly inappropriate in a particular case). One idea suggested at IMO 2015, to allow the desired proportions to be attained more exactly, was to introduce tie-breaking rules between contestants on the same score (for example, based on the sum of squares of individual problem scores). Such suggestions are not covered in this analysis, but even if used, algorithms like those here would still be relevant because of students at the boundaries with the same multiset of scores (or if applicable with the rule in question, the same scores on every problem).

Some other mathematical olympiads also decide medal boundaries following similar rules to the IMO.

Bronze medal boundary

The algorithms here relate to choosing the total number of contestants to receive a medal. All these algorithms choose between two possibilities: the closest numbers above and below half the number of contestants. (If it is possible for exactly half the contestants to receive a medal, there is only one possibility.)

The following table shows the ideal number of medals each year and the choices for how many medals to go below or above that number, with the choice made in bold.

Year	Ideal	Below	Above
1986	105	3	**2**
1987	$118\frac{1}{2}$	$3\frac{1}{2}$	$\mathbf{1\frac{1}{2}}$
1988	134	**4**	5
1989	$145\frac{1}{2}$	$5\frac{1}{2}$	$\mathbf{1\frac{1}{2}}$
1990	154	16	**1**
1991	156	**1**	5
1992	$175\frac{1}{2}$	$\mathbf{5\frac{1}{2}}$	$9\frac{1}{2}$
1993	$206\frac{1}{2}$	$\mathbf{8\frac{1}{2}}$	$2\frac{1}{2}$
1994	$192\frac{1}{2}$	$\mathbf{\frac{1}{2}}$	$11\frac{1}{2}$
1995	206	**5**	3
1996	212	**12**	3
1997	230	10	**1**
1998	$209\frac{1}{2}$	$\mathbf{4\frac{1}{2}}$	$7\frac{1}{2}$
1999	225	24	**1**
2000	$230\frac{1}{2}$	$\mathbf{1\frac{1}{2}}$	$28\frac{1}{2}$
2001	$236\frac{1}{2}$	$9\frac{1}{2}$	$\mathbf{5\frac{1}{2}}$
2002	$239\frac{1}{2}$	$\mathbf{7\frac{1}{2}}$	$8\frac{1}{2}$
2003	$228\frac{1}{2}$	$\mathbf{18\frac{1}{2}}$	$1\frac{1}{2}$
2004	243	**0**	**0**
2005	$255\frac{1}{2}$	$\mathbf{7\frac{1}{2}}$	$4\frac{1}{2}$
2006	249	61	**4**
2007	260	**7**	5
2008	$267\frac{1}{2}$	$\mathbf{\frac{1}{2}}$	$15\frac{1}{2}$
2009	$282\frac{1}{2}$	$\mathbf{\frac{1}{2}}$	$12\frac{1}{2}$
2010	$258\frac{1}{2}$	$32\frac{1}{2}$	$\mathbf{7\frac{1}{2}}$
2011	282	**1**	20
2012	274	20	**3**
2013	$263\frac{1}{2}$	$14\frac{1}{2}$	$\mathbf{14\frac{1}{2}}$
2014	280	14	**15**
2015	$288\frac{1}{2}$	$\mathbf{6\frac{1}{2}}$	$17\frac{1}{2}$
2016	301	**21**	10
2017	$307\frac{1}{2}$	$\mathbf{16\frac{1}{2}}$	$33\frac{1}{2}$

The following list shows which years from 1986 onwards various algorithms would have failed to predict the bronze medal boundary correctly (the arbitrary nature of some of the algorithms is because they are chosen to model many past choices, rather than on any theoretical basis). Algorithms are described in terms of a choice between going a medals above half the number of contestants and going b medals below; after the first two algorithms listed, they attempt to model various notions of an unusual distribution of marks that might justify going over half the number of contestants.

- Never give medals to more than half the contestants: : 1986, 1987, 1989, 1990, 1997, 1999, 2001, 2006, 2010, 2012, 2013, 2014.
- Give as near half the contestants as possible medals, being generous in case of equality ($b \geq a$): 1993, 1995, 1996, 2003, 2005, 2007, 2014, 2016.
- Go over if $b \geq 5a$: 1986, 1987, 1989, 2001, 2003, 2010, 2013, 2014.
- Go over if $b > 4a$: 1986, 1987, 1989, 2001, 2003, 2013, 2014.
- Go over if $b \geq 3.5a$: 1986, 1987, 1996, 2001, 2003, 2013, 2014.
- Go over if $b \geq 1.5a^2$: 1986, 2001, 2003, 2010, 2013, 2014.

Gold and silver medal boundaries

The algorithms here all work based on a previously determined bronze medal boundary; for the analysis here, this is the boundary actually chosen by the Jury. This is to accord most closely with historical practice, although of course algorithms could be adapted to determine all three boundaries together in a similar way.

As there are two boundaries to be chosen here, to approximate a ratio 1 : 2 : 3, there may not necessarily be only two obvious choices in all cases. We consider the following algorithms (in all cases, ties are broken by being generous—first by being generous regarding the total number of gold and silver medals, then by being generous regarding the number of gold medals). Each algorithm is given a code used in the following table.

- Boundaries may be chosen independently, to make the number of gold and silver medals as close as possible to half the number of medals, and to make the number of gold medals as close as possible to one sixth of the number of medals. (I)
- Or the silver boundary may be chosen first, to make the number of gold and silver medals as close as possible to half the number of medals, followed by choosing the gold boundary to make the number of gold medals as close as possible to a third of the number of gold and silver medals. (S)
- Or we may try to minimise the total deviation of the numbers of each type of medal from the ideal by looking at the L^p norm of the vector of deviations from the ideal; or we may do this after scaling the numbers of each type of medal so that all the ideal numbers are equal. (L^p, $L^p s$)
- Finally, we consider minimising a deviation determined directly from the ratios: $2g/s + s/2g + 3g/b + b/3g + 3s/2b + 2b/3s \geq 6$ by AM-GM, with equality when the medals achieve the ideal proportions. (This is my preferred approach.) (R)

The following table shows when each of these approaches has failed to predict the boundaries chosen by the Jury. It will be seen that these algorithms are a less good match before around 2000, with the Jury apparently having commonly chosen to be less generous about these choices (fewer gold medals or more bronze medals) before then. The boundaries actually chosen are in bold; other choices of boundaries have the algorithms that would have chosen them indicated. Stricter choices of boundaries (fewer gold and silver medals, then fewer gold medals) are on the left.

1986	(18, 35, 54) $ISL^1L^2L^1sL^2sR$	**(18, 41, 48)**	
1987	(22, 36, 62) IS	**(22, 42, 56)**	
1988	**(17, 48, 65)**	(24, 41, 65) $ISL^1L^2L^1sL^2sR$	
1989	**(20, 55, 72)**	(27, 48, 72) $ISL^1L^2L^1sL^2sR$	
1990	**(23, 56, 76)**	(28, 51, 76) $ISL^1L^2L^1sL^2sR$	
1991	**(20, 51, 84)**	(20, 61, 74) I	(33, 48, 74) S
1992	**(26, 57, 87)**	(26, 61, 83) I	(32, 55, 83) S
1993	**(35, 66, 97)**		
1994	**(30, 64, 98)**	(33, 61, 98) IL^2sR	
1995	**(30, 71, 100)**	(35, 66, 100) $ISL^1L^2L^1sL^2sR$	
1996	(32, 69, 99) I	**(35, 66, 99)**	
1997	**(39, 70, 122)**	(39, 80, 112) $ISL^1L^2L^1sL^2sR$	
1998	**(37, 66, 102)**		
1999	(36, 72, 118) L^2R	**(38, 70, 118)**	(38, 80, 108) $ISL^1L^1sL^2s$
2000	(35, 75, 119) SL^1L^2	**(39, 71, 119)**	
2001	**(39, 81, 122)**		
2002	**(39, 73, 120)**		
2003	**(37, 69, 104)**		
2004	(37, 86, 120) IL^2sR	**(45, 78, 120)**	
2005	**(42, 79, 127)**		
2006	**(42, 89, 122)**	(44, 87, 122) $SL^1L^2L^2sR$	
2007	**(39, 83, 131)**		
2008	(42, 84, 141) $SL^1L^2L^1sL^2sR$	(47, 79, 141) I	**(47, 100, 120)**
2009	**(49, 98, 135)**	(53, 94, 135) L^1	
2010	**(47, 104, 115)**	(52, 99, 115) SL^2	(59, 92, 115) L^1
2011	**(54, 90, 137)**		
2012	**(51, 88, 138)**		
2013	**(45, 92, 141)**		
2014	**(49, 113, 133)**		
2015	**(39, 100, 143)**	(54, 85, 143) IL^2sR	
2016	(44, 92, 144) $ISL^1L^2L^1sL^2sR$	**(44, 101, 135)**	
2017	**(48, 90, 153)**		

References

All the code used to implement different algorithms for the above analyses is available with git at:

```
https://git.ukmt.org.uk/medal-boundaries.git
git://git.ukmt.org.uk/git/medal-boundaries.git
```

There is also a mirror on GitHub:

```
https://github.com/jsm28/medal-boundaries
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This analysis document may be revised from time to time. See that repository for details of previous versions.

The code downloads data from imo-official. Note that the ideal number of medals is half the number of non-disqualified contestants (not half a total number that includes disqualified contestants, in 1991 and 2010), and in 2005 all calculations are done without including two contestants to whom a translated paper was transmitted inaccurately (because that was the basis on which boundaries were determined that year, with those contestants then being given prizes as if they had scored 7 on the affected problem), meaning that figures differ slightly from those appearing on imo-official if this is not allowed for. Also note that the calculations here were done with data from before the disqualification in 2016 of an ineligible student from 2010, 2011 and 2012, reflecting the scores available to the Jury at the time they made their decisions (that student having been given, before disqualification, scores of 21 in 2010, 14 in 2011 and 18 in 2012) (but the code does not currently make the adjustments for that case itself, as it does for the 2005 case).

The code is written for simplicity rather than efficiency (for example, testing all possibilities for gold and silver boundaries rather than more carefully bounding what cases need to be tested to find the optimum).