C6 with general initial configuration

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In the mark scheme discussion for problem 3 (C6) it was indicated that reductions from ‘useful’ initial configurations (not restricted to the precise one in the official solution) would be worth a mark but those from ‘random’ initial configurations wouldn’t. This solution illustrates that arbitrary initial configurations (without any reference to a particular maximum clique) are in fact useful and so may also need to be credited for consistency of marking of partial solutions between the different approaches.

Let G be the graph of all competitors, and let $c(H)$ be the largest size of a clique in H (for H a subgraph or subset of vertices of G); let $c(G) = 2m$. Suppose that G is a counterexample to the problem, i.e., that its vertices cannot be divided into two parts with equal largest clique size.

Starting from an arbitrary division of the vertices of G into G_1 and G_2, move vertices from the part with the greater largest size of a clique into the other part (as in the official solution) until the sizes differ by 1, say wlog $c(G_1) = r$ and $c(G_2) = r + 1$; as in the official solution, $r \geq m$. We may suppose r maximum such that there exists such a division; then there do not exist two vertex-disjoint cliques of size $r + 1$.

Lemma: G_2 contains a unique clique of size $r + 1$.

Proof: Suppose otherwise; let U be the smallest union of the sets of vertices of two K_{r+1} in G_2. Move vertices contained in a K_{r+1} in G_2 but not in U into G_1 one-by-one; since we have a counterexample, this preserves $c(G_1)$ and $c(G_2)$. Now let H_1, H_2 be two distinct K_{r+1} with vertices in U, and let $a \in H_1 \setminus H_2$, $b \in H_2 \setminus H_1$ be two vertices in U; then any K_{r+1} in G_2 contains at least one of a and b (by minimality of U). $c(G_2 - a) = c(G_2 - b) = r + 1$ so $c(G_1 + a) = c(G_1 + b) = r$, but $c(G_2 - a - b) = r$ so $c(G_1 + a + b) = r + 1$, and any K_{r+1} in $G_1 + a + b$ must contain both a and b, so ab is an edge. Since a and b were arbitrary vertices in $H_1 \setminus H_2$ and $H_2 \setminus H_1$, the vertices of U form a clique, which has size greater than $r + 1$, a contradiction. □

Proof of C6: Now G_2 contains a unique clique of size $r + 1$. Moving any vertex a_i of that clique to G_1 yields a unique clique $H_i + a_i$ of size $r+1$ in $G_1 + a_i$, and not all H_i are the same K_r subgraph (else we have a clique of size $2r + 1$ in G), so say $H_1 \neq H_2$, $b_1 \in H_1 \setminus H_2$ and $b_2 \in H_2 \setminus H_1$. Then $G_2 - a_1 + b_1$ and $G_2 - a_2 + b_2$ contain cliques of size $r + 1$ (containing b_1 and b_2 respectively). The clique in $G_2 - a_1 + b_1$ must contain a_2, since otherwise it would be disjoint from $H_2 + a_2$, so $b_1 a_2$ is an edge. Since b_1 was an arbitrary vertex of $H_1 \setminus H_2$, a_2 has edges to all vertices of H_1, so $G_1 + a_2$ has more than one clique of size $r + 1$, contradicting the lemma. □

(The Lemma may also be applied to the result of Step 2 of the official solution, where G_1 is a clique of size r that must then have all its vertices joined to all the vertices of the unique K_{r+1} in G_2.)